Finite p-groups of class two with a very large multiple holomorph

Cindy Tsang[1]

Joint work with Andrea Caranti[2]

Ochanomizu University (Japan)

University of Trento (Italy)

Hopf Algebras & Galois Module Theory

Online & University of Nebraska Omaha

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[2]Member of GNSAGA-INdAM, Italy

Holomorph

- Let G be a group.
- The (abstract) holomorph of G is defined to be

$$\mathrm{Hol}(G) = G \rtimes \mathrm{Aut}(G) \quad \text{(outer semidirect product)}.$$

- Let $\mathrm{Perm}(G)$ be the group of all permutations on G.
- One can identify G as a subgroup of $\mathrm{Perm}(G)$ via
  1. the left regular representation: $\lambda(\sigma) = (x \mapsto \sigma x)$ for each $\sigma \in G$
  2. the right regular representation: $\rho(\sigma) = (x \mapsto x\sigma^{-1})$ for each $\sigma \in G$
- The (permutational) holomorph of G is defined to be

$$\mathrm{Hol}(G) = \lambda(G) \rtimes \mathrm{Aut}(G) \quad \text{(inner semidirect product)}$$
$$\mathrm{Hol}(G) = \rho(G) \rtimes \mathrm{Aut}(G) \quad \text{(inner semidirect product)}$$

  regarded as a subgroup of $\mathrm{Perm}(G)$.
- The abstract and permutational holomorphs are isomorphic.

Multiple holomorph

- Let G be a group.
- Consider its permutational holomorph

$$\mathrm{Hol}(G) = \lambda(G) \rtimes \mathrm{Aut}(G) = \rho(G) \rtimes \mathrm{Aut}(G).$$

- It is easy to show that

$$\mathrm{Hol}(G) = \mathrm{Norm}_{\mathrm{Perm}(G)}(\lambda(G)) = \mathrm{Norm}_{\mathrm{Perm}(G)}(\rho(G))$$

- The multiple holomorph of G is defined to be

$$\begin{aligned} \mathrm{NHol}(G) &= \mathrm{Norm}_{\mathrm{Perm}(G)}(\mathrm{Hol}(G)) \\ &= \mathrm{Norm}_{\mathrm{Perm}(G)}(\mathrm{Norm}_{\mathrm{Perm}(G)}(\lambda(G))) \\ &= \mathrm{Norm}_{\mathrm{Perm}(G)}(\mathrm{Norm}_{\mathrm{Perm}(G)}(\rho(G))) \end{aligned}$$

- We are interested in the quotient group

$$T(G) = \mathrm{NHol}(G)/\mathrm{Hol}(G)$$

of the multiple holomorph by the holomorph.

Connection with regular subgroups

- Isomorphic regular subgroups of $\mathrm{Perm}(G)$ are conjugates of each other.
- The regular subgroups of $\mathrm{Perm}(G)$ which are isomorphic to G are therefore precisely the conjugates of $\rho(G)$. Say $N = \pi^{-1}\rho(G)\pi$, where $\pi \in \mathrm{Perm}(G)$.
  - $\boxed{\pi \in \mathrm{NHol}(G) \iff \mathrm{Norm}_{\mathrm{Perm}(G)}(N) = \mathrm{Hol}(G)}$ So restricting to $\mathrm{NHol}(G)$ means that we only consider the regular subgroups whose normalizer is also equal to $\mathrm{Hol}(G)$.
  - $\boxed{\pi \in \mathrm{Hol}(G) \iff N = \rho(G)}$ So modding out by $\mathrm{Hol}(G)$ means that we get each regular subgroup exactly once.
- Thus, the quotient group $T(G)$ acts regularly via conjugation on ...
  - the regular subgroups of $\mathrm{Perm}(G)$ which are isomorphic to G and whose normalizer is also equal to $\mathrm{Hol}(G)$.
- In the case that G is finite, these are just ...
  - the normal regular subgroups of $\mathrm{Hol}(G)$ which are isomorphic to G.

Some consequences of $T(G)$ being large

- In the case that G is finite, the quotient group $T(G)$ parametrizes normal regular subgroups N of $\mathrm{Hol}(G)$ which are isomorphic to G.
- Any such regular subgroups N_1, N_2 normalize each other.
- In relation to skew braces ...
  - N_1, N_2 yield a bi-skew brace on the set G.
  - *Both the additive and multiplicative groups are isomorphic to G.
  - The operations on G coming from these N's form a brace block.

  Thus, if $T(G)$ is very large, then we get very large brace block.
- In relation to normalizing graphs ...
  - N_1, N_2 are connected by an edge in the normalizing graph of G.
  - The edges joining these vertices N's form a complete subgraph.

  Thus, if $T(G)$ is very large, then we get a very large clique.

Timeline of the study of $T(G)$

- 1908: G. A. Miller — finite abelian groups
- 1951: W. H. Mills — finitely generated abelian groups
- 2015: T. Kohl — dihedral and dicyclic groups
- 2018: A. Caranti & F. Dalla Volta — finite centerless perfect groups
- 2019: T. — finite almost simple groups
- 2020: T. — groups of squarefree orders
- arXiv: T. — certain centerless groups
- In all of the above cases $T(G)$ turns out to be elementary 2-abelian.
- But there are also many examples where $T(G)$ is not elementary 2-abelian.
- 2018: A. Caranti — certain finite p-groups of class 2
- 2022: T. — certain finite split metacyclic p-groups
- Under suitable conditions, the order of $T(G)$ is divisible by $p-1$ or p.

Main result

- Consider finite p-groups G with p an odd prime.
- Question. Can the order of $T(G)$ have divisors outside of $p-1$ and p?
- Answer. Yes. In fact, the order of $T(G)$ can be made very large.

Main Theorem (A. Caranti & T., arXiv:2205.15205)

For any $n \geq 4$, there is a finite p-group G of class two of order $p^{n+\binom{n}{2}}$ such that

$$T(G) \simeq \mathbb{F}_p^{\binom{n}{2}\binom{n+1}{2}} \rtimes \left(\mathbb{F}_p^{(\binom{n}{2}-n)\times n} \rtimes \left(\mathrm{GL}_n(\mathbb{F}_p) \times \mathrm{GL}_{\binom{n}{2}-n}(\mathbb{F}_p) \right) \right),$$

where the second semidirect product is given by $Q^{(A,M)} = M^{-1}QA$.

- Since every finite group embeds into $\mathrm{GL}_N(\mathbb{F}_p)$ for N large enough...

Corollary (A. Caranti & T., arXiv:2205.15205)

For any finite group H and any sufficiently large $n \geq 4$, there is a finite p-group G of class two of order $p^{n+\binom{n}{2}}$ such that H embeds into $T(G)$.

The idea of the proof

Keep in mind that in the case that G is finite, the quotient group $T(G)$ parametrizes normal regular subgroups of $\mathrm{Hol}(G)$ which are isomorphic to G.

Multiple holomorph and bilinear forms

- Consider finite p-groups G of class two with p an odd prime.

Theorem (A. Caranti, 2018)

There is a bijection between:

(a) the normal regular subgroups of $\mathrm{Hol}(G)$ whose projection onto $\mathrm{Aut}(G)$ lies in

$$\underbrace{\mathrm{Aut}_c(G) \cap \mathrm{Aut}_z(G)}_{\text{the subgroup consisting of all } \varphi \in \mathrm{Aut}(G) \text{ which induces the identity on } G/Z(G) \text{ and } Z(G)}$$

(b) the bilinear forms $\Delta : G/Z(G) \times G/G' \longrightarrow Z(G)$ which satisfy

$$\Delta(x^\varphi, y^\varphi) = \Delta(x, y)^\varphi \text{ for all } x, y \in G \text{ and } \varphi \in \mathrm{Aut}(G)$$

- Given such a bilinear form Δ, the corresponding normal regular subgroups is isomorphic to $(G, \circ)$, where $\circ$ is explicitly defined by

$$x \circ y = xy\Delta(x, y) \text{ for all } x, y \in G.$$

- We need to know when this circle group $(G, \circ)$ is actually isomorphic to G.

Multiple holomorph and bilinear forms

- Let us restrict to finite p-groups G of class two such that

$$G' = Z(G),\ \mathrm{Aut}(G) = \mathrm{Aut}_c(G),\ \text{which imply } \mathrm{Aut}(G) = \mathrm{Aut}_z(G).$$

The conditions on the regular subgroups and bilinear forms become vacuous.

Theorem (A. Caranti, 2018)

There is a bijection between:

a. the normal regular subgroups of $\mathrm{Hol}(G)$ ~~whose projection onto $\mathrm{Aut}(G)$ lies in~~

$$\sout{\underbrace{\mathrm{Aut}_c(G) \cap \mathrm{Aut}_z(G)}_{\text{the subgroup consisting of all } \varphi \in \mathrm{Aut}(G) \text{ which induces the identity on } G/Z(G) \text{ and } Z(G)}}$$

b. the bilinear forms $\Delta : G/Z(G) \times G/G' \longrightarrow Z(G)$ ~~which satisfy~~

$$\sout{\Delta(x^{\varphi}, y^{\varphi}) = \Delta(x, y)^{\varphi} \text{ for all } x, y \in G \text{ and } \varphi \in \mathrm{Aut}(G)}$$

- In this case, the quotient group $T(G)$ parametrizes such bilinear forms Δ whose circle group $(G, \circ)$, defined by $x \circ y = xy\Delta(x, y)$, is isomorphic to G.

Basic set-up

- Consider finite p-groups G of class two such that

$$G' = Z(G),\ \mathrm{Aut}(G) = \mathrm{Aut}_c(G),\ \text{which imply } \mathrm{Aut}(G) = \mathrm{Aut}_z(G).$$

  The previous theorem implies that there is bijection between:

  a. the normal regular subgroups of $\mathrm{Hol}(G)$
  b. the bilinear forms $\Delta : G/G' \times G/G' \longrightarrow G'$

  We need to know when $(G, \circ)$ is isomorphic to G, where $x \circ y = xy\Delta(x, y)$.

- In this case, since both of the arguments come from the same group G/G', we can define the following notion:

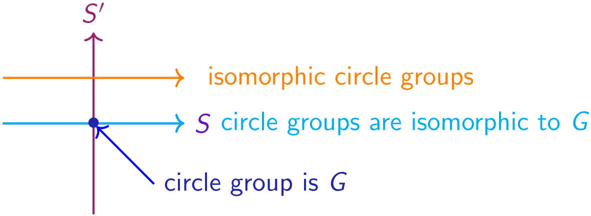
  1. Δ is symmetric $\iff \Delta(y, x) = \Delta(x, y)$ for all $x, y \in G$
  2. Δ is anti-symmetric $\iff \Delta(y, x) = \Delta(x, y)^{-1}$ for all $x, y \in G$

- Moreover, every bilinear form Δ decomposes as a product of a symmetric form with an anti-symmetric form via

$$\Delta(x, y) = (\Delta(x, y)\Delta(y, x))^{\frac{1}{2}} \cdot \left(\Delta(x, y)\Delta(y, x)^{-1}\right)^{\frac{1}{2}}.$$

Isomorphism class of the circle groups

- The set of all bilinear forms is a group under multiplication in G'.
- Let S denote the subgroup consisting of all symmetric forms.
- Let S' denote the subgroup consisting of all anti-symmetric forms.



- Every bilinear form may be regarded as a dot on this plane.
- The origin corresponds to the trivial bilinear form $\Delta_0(x, y) \equiv 1$, and the corresponding circle group $(G, \circ)$ is equal to G because then $x \circ y = xy$.

Proposition (A. Caranti & T., arXiv:2205.15205)

Two points lying on the same horizontal slice have isomorphic circle groups.

Circle groups coming from symmetric forms

Corollary (A. Caranti & T., arXiv:2205.15205)

Circle groups coming from symmetric forms are isomorphic to G.

- Direct proof. Let $\Delta : G/G' \times G/G' \to G'$ be a symmetric form. Let

$$x \circ y = xy\Delta(x, y)$$

denote its circle group on G. Then

$$\theta : G \to (G, \circ); \;\; x^\theta = x\Delta(x, x)^{1/2}$$

is an isomorphism. It is a homomorphism because

$$\begin{aligned}(xy)^\theta &= xy\Delta(xy, xy)^{1/2} \\ &= xy\Delta(x, x)^{1/2}\Delta(y, y)^{1/2}\Delta(x, y) \\ &= x\Delta(x, x)^{1/2} \circ y\Delta(y, y)^{1/2} \\ &= x^\theta \circ y^\theta\end{aligned}$$

It is injective because $x^\theta = 1$ implies $x \in G'$ and θ is the identity on G'.

It is then surjective because G and $(G, \circ)$ have the same finite order. □

Symmetric forms vs Anti-symmetric forms

- The anti-symmetric forms are much harder than the symmetric forms.

- Circle groups coming from symmetric forms are isomorphic to G.
- Not the case for anti-symmetric forms.

- The set of bilinear forms is a group under multiplication in G'.
- Let Δ_1, Δ_2 be such that their circle groups are both isomorphic to G, so they correspond to some elements $\theta_1\mathrm{Hol}(G), \theta_2\mathrm{Hol}(G)$ in $T(G)$.
- If Δ_1, Δ_2 are both symmetric, then $\Delta_1\Delta_2$ corresponds to $\theta_1\theta_2\mathrm{Hol}(G)$, that is, the group operations of symmetric forms and $T(G)$ agree.
- Not the case for anti-symmetric forms.

- We looked at a special family of groups for which G/G' and G' are both elementary abelian so that we can use linear algebra.

Main result (revisited)

- Consider finite p-groups G with p an odd prime.
- Question. Can $T(G)$ have divisors outside of $p-1$ and p?
- Answer. Yes. In fact, the order of $T(G)$ can be made very large.

Main Theorem (A. Caranti & T., arXiv:2205.15205)

For any $n \geq 4$, there is a finite p-group G of class two of order $p^{n+\binom{n}{2}}$ such that

$$T(G) \simeq \underbrace{\mathbb{F}_p^{\binom{n}{2}\binom{n+1}{2}}}_{\text{from symmetric forms}} \rtimes \underbrace{\left(\mathbb{F}_p^{(\binom{n}{2}-n)\times n} \rtimes \left(\mathrm{GL}_n(\mathbb{F}_p) \times \mathrm{GL}_{\binom{n}{2}-n}(\mathbb{F}_p)\right)\right)}_{\text{from anti-symmetric forms}},$$

where the second semidirect product is given by $Q^{(A,M)} = M^{-1}QA$.

- The symmetric part is elementary abelian and has very simple structure.
- The anti-symmetric part is much more complicated.

Thank you for listening!